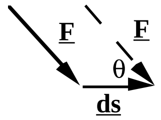
# Work and Energy




When force varies, use differential displacement **ds**

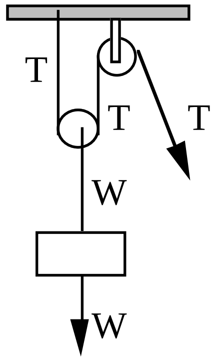
$$dW \equiv F\,ds\cos\theta$$

*ds X component of F // ds*
*or*
*F X component of ds // F*

$$W = \int_0^L F\cos\theta\,ds$$

*if F is constant*, we get $W = FL\cos\theta$

SI Unit: 1 Newton x 1 metre = 1 Joule




If the rope and pulleys are light, and if the acclerations are negligible, then

Force on LH pulley

$ma \cong 0 = 2T - W$

$\therefore\ T = W/2$

If mass rises by D, word done = WD.
But rope shortens on both sides of rising pulley,
if mass rises by D, rope must be pulled 2D, so work done = T X 2D = WD

**Example**. $F_{grav} \propto 1/r^2$. How much work is done to move m = 1 tonne from earth's surface (r = 6500 km) to r = ∞?

$$W = \int F\,ds\cos\theta$$

$$= \int F\,dr$$

$$F = -F_{grav} = \frac{Cm}{r^2}$$

On surface F/m = 9.8 ms$^{-2}$

$\therefore\ C = (9.8\ ms^{-2})(6.5\ 10^6\ m)^2 = 4.1\ 10^{14}\ m^3s^{-2}$

$$W = \int_{6500\ km}^{\infty} \frac{Cm}{r^2}\,dr$$

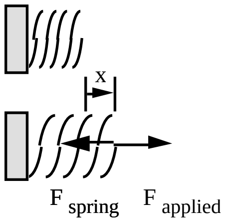
$$= -Cm\left(\frac{1}{\infty} - \frac{1}{6.5\ 10^6}\right)$$

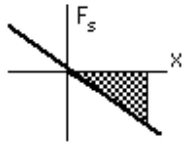
$$=$$

$$= 6.3\ 10^{10}\ J \quad = 63\ GJ.$$

**Work to deform spring**

No applied force
(x = 0)




Work done **by** spring = $\int F_{spring}.dx$

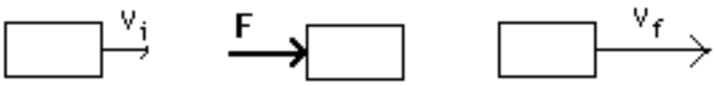
$= \int -kx.dx = -\frac{1}{2}kx^2$

Work done **on** spring = $\int F_{applied}.dx$

$= \int kx.dx = +\frac{1}{2}kx^2$

**Work and kinetic energy**

(Total) force F acts on mass m in x direction.




Work done by F = $\int_i^f F\,dx$ *(use F = ma)*

$= \int_i^f m\frac{dv}{dt}dx = \int_i^f m\frac{dx}{dt}dv$

$= \int_i^f mv.dv$

Work done by F = $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

Define **kinetic energy** $K \equiv \frac{1}{2}mv^2$

Increase in kinetic energy of body = work done by **total** force acting on it.

**Power**. Rate of doing work

Average power $\bar{P} \equiv \frac{W}{\Delta t}$

Instantaneous power $P = \frac{dW}{dt}$

SI unit: 1 Joule per second ≡ 1 Watt (1 W)

*(humans can produce 100s of W, car engines several tens of kW)*

**Potential energy**.

e.g. Compress **spring**, do W on it, but get no K. Yet can get energy out: spring can expand and give K to a mass. → Idea of stored energy.

e.g. **Gravity**: lift object (slowly), do work but get no K. Yet object can fall back down and get K.

**But:** Slide mass slowly along a surface. Do work against **friction**, but can't recover this energy mechanically. Not all forces "store" energy

**Conservative and non-conservative forces**

$$W_{against\ grav} = -\int_i^f F_g dr \cos\theta$$

$$= -\int_i^f F_g\ dz$$

$$= mg\int_i^f dz$$

$$= mg(z_f - z_i) \qquad \textit{in uniform field}$$

W is uniquely defined at all **r**, i.e. W = W(**r**)
If $z_f - z_i$ are the same, W = 0.

∴ Work done against gravity round a closed path = 0
Gravity is a **conservative force**

$$W_{against\ spring} = -\int_i^f F_{spring}.dx$$

$$= -\int_i^f -kx.dx = \frac{1}{2}k(x_f^2 - x_i^2)$$

W is uniquely defined at all x, i.e. W = W(x)

$x_f = x_i \Rightarrow W = 0.$

∴ Work done round a closed path = 0

Spring force is a **conservative force**

**Friction**

$dW_{against\ fric} = -F_f\ ds \cos\theta$
but $F_f$ always has a component *opposite* ds

∴ dW always ≥ 0. *(we never get work back)*

∴ cannot be zero round closed path, ∴ W ≠ W(**r**)

∴ friction is a **non-conservative force**

**Potential energy**

For a **conservative** force **F** (i.e. one where work done against it, W = W(**r**)) we can define potential energy U by $\Delta U = W_{against}$. i.e.

$$\Delta U = -\int_i^f F\ dr \cos\theta$$

Same examples: **spring**

$$\Delta U_{spring} = -\int_i^f F_{spring}.dx = \frac{1}{2}k(x_f^2 - x_i^2)$$

**Choice of zero for U is arbitrary.**

Here U = 0 at x = 0 is obvious, so $U_{spring} = \frac{1}{2}kx^2$

**From energy to force:**

$U = -\int F\ ds$ where ds is in the direction // F

$$F = -\frac{dU}{ds}$$

*in fact* $F_x = -\frac{dU}{dx}, F_y = -\frac{dU}{dy}, F_z = -\frac{dU}{dz}$

Spring: $U_{spring} = \frac{1}{2}kx^2 \therefore F_{spring} = -kx$

Gravity: $U_g = mgz \therefore F_g = -\frac{dU}{dz} = -mg$

Energy of interaction:

**Conservation of mechanical energy**

Recall: Increase in K of body = work done by **total** force acting on it. *(restatement of Newton 2)*

But, if all forces are conservative, work done **by** these forces $= -\Delta U$ *(definition of U)*

$\therefore$ if only conservative forces act, $\Delta K = -\Delta U$

We define mechanical energy

$$E \equiv K + U$$

so, if only conservative forces act, $\Delta E = 0$.

*we can make this stronger.*

Work done by **non-conservative forces**

Define internal energy $U_{int}$ where

$\Delta U_{int} = -$ Work done **by** n-c forces

(= + Work done **against** n-c forces)

Recall def$^n$ of K: $\Delta K$ = work done by $\Sigma$ force

$$\therefore \quad \Delta K = -\Delta U - \Delta U_{int}$$

$$\therefore \quad \Delta K + \Delta U + \Delta U_{int} = 0$$

If n-c forces do no work, then $\Delta U_{int} = 0$, so:

**If non-conservative forces do no work,**

$$\Delta E = \Delta K + \Delta U = 0$$

or: **mechanical energy is conserved**

***Never, ever write***: *"kinetic energy = potential energy"*

**Example.** Freda (m = 60 kg) rides pogo stick (m << 60 kg) with spring constant k = 100 kN.m$^{-1}$. Neglecting friction, how far does spring compress if jumps are 50 cm high?

Non-conservative forces do no work, $\therefore$ mechanical energy is conserved, i.e.

$E_{bottom} = E_{top}$

$K_b + U_b = K_t + U_t$ *($U = U_{grav} + U_{spring}$)*

$$\frac{1}{2}mv_{horiz}^2 + (mgy_b + \frac{1}{2}kx_b^2) \cong \frac{1}{2}mv_{horiz}^2 + (mgy_t + \frac{1}{2}kx_t^2)$$

$$mg(y_t - y_b) \cong \frac{1}{2}kx_b^2$$

$$\therefore \quad x_b \cong \sqrt{\frac{2mg(y_t - y_b)}{k}} \cong 80 \text{ mm.}$$