Question 1 [30 marks] Griffiths 4.15. A thick spherical shell (inner radius = a, outer radius = b) is made of a dielectric material with a "frozen-in" polarization

\[ P(r) = \frac{k}{r} \hat{r} \]

where \( k \) is a constant and \( r \) is the distance from the centre. (Note: there is no free charge in the problem). You have to find the electric field \( \mathbf{E} \) in all three regions i.e. \( r < a; a < r < b \) and \( r > b \) by two different methods. Express your answers in terms of \( k \) and \( r \) and any appropriate constants or other factors. Thus,

(a) Locate all the bound charge and then use Gauss' law to calculate the field it produces, in all three regions.
(b) Find \( \mathbf{D} \) and then get \( \mathbf{E} \), in all three regions.

a) \[ \rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{k}{r} \right) = -\frac{k}{r^2} \, \text{Cm}^{-3} \]

\[ \sigma_b = \mathbf{P} \cdot \hat{n} \]

inner surface \[ \mathbf{P} = \frac{k}{a} \hat{r} \]

\[ \sigma_b = -\frac{k}{a} \, \text{Cm}^{-2} \]

outer surface \[ \mathbf{P} = \frac{k}{b} \hat{r} \]

\[ \sigma_b = +\frac{k}{b} \, \text{Cm}^{-2} \]
\[
\text{Gauss}\rightarrow E = \frac{Q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \quad r < a
\]

\[
 r < a \quad Q_{\text{enc}} = 0 \quad \Rightarrow \quad E = 0
\]

\[
 r > b \quad Q_{\text{enc}} = 0 \quad \text{(neutral dielectric)} \quad \Rightarrow \quad E = 0
\]

\[
a \leq r \leq b
\]

\[
Q_{\text{enc}} = \text{surface charge at } r = c a
\]

\[
\text{(inner surface)}
\]

\[
+ \text{ volume charge to } r
\]

\[
Q_{\text{surface}} = 6_b \cdot A_{\text{surf}} = \left(-\frac{k}{a}\right)(4\pi a^2)
\]

\[
= -4\pi ka \quad C
\]
\[ Q_{\text{vol}} = \int \rho \, d\tau \]
\[ = -\iiint \frac{k}{r^2} \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi \]
\[ = -4\pi k \int_a^r \, dr = -4\pi k (r-a) \, C \]

\[ Q_{\text{enc}} = -4\pi k a - 4\pi k (r-a) \]
\[ = -4\pi k r \]

\[ E = -\frac{4\pi k r}{4\pi \varepsilon_0 r^2} \hat{r} = -\frac{k}{\varepsilon_0 r} \hat{r} \]

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b) \( \mathbf{D} = 0 \) everywhere because there is no free charge anywhere in this problem.

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = 0 \]

\[ \therefore \mathbf{E} = -\frac{\mathbf{P}}{\varepsilon_0} \quad \text{which gives the same results as in part (a)} \]
Question 2 [10 marks]

The electric displacement $D$ in a vacuum is given by $D = xi + y^2j$ $C/m^2$ Calculate the corresponding volume charge density in $C/m^3$ at the point $(x=2, y=3, z=4)$.

$$\nabla \cdot D = 1 + 2y = \rho$$

$$\therefore \rho(2,3,4) = 7 \text{ C/m}^3$$

Question 3 [10 marks]

A 10 nC point charge is located at the origin in free space.

(a) Calculate $D$ and $E$ at a distance 1 m from the origin.

(b) Repeat part (a) but this time with the charge located in a large volume of distilled water (dielectric constant = 81).

(a) \[ \int \mathbf{D} \cdot d\mathbf{a} = Q_{\text{free}} \]

\[ 4\pi r^2 \mathbf{D} = Q = 10 \text{ nC} \]

\[ \mathbf{D} = \frac{10 \text{ nC}}{4\pi \text{ m}^2} \]

\[ \mathbf{D} = 0.796 \hat{r} \text{ nC m}^2 \]

\[ E = \frac{\mathbf{D}}{\varepsilon_0} = 89.9 \hat{r} \text{ V/m} \]

(b) Repeating part (a) with the charge in a large volume of distilled water.

(b) $D$ is unchanged, so the free charge is unchanged.

$$D = \varepsilon_0 E + P = k\varepsilon_0 E$$

$$E = \frac{D}{k\varepsilon_0} = \frac{E_{\text{original}}}{k}$$

$$= \frac{89.9}{81}$$

$$= 1.11 \text{ Vm}^{-1}$$